ASYMPTOTIC BEHAVIOR OF THE SOLUTIONS OF THE DIFFERENCE EQUATION $x_{n+1} = x_n^2 f(x_{n-1})$

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ABSTRACT. The difference equation

$$x_{n+1} = x_n^2 f(x_{n-1}),$$

where f is a nonincreasing real valued function, is discussed and asymptotic properties of the (nonnegative) solutions are provided.

1. Introduction

Motivated by an open problem presented in the monograph of Kocic and Ladas [2, p. 159], recently D.C.Zhang, B. Shi and M.J. Gai [1] investigated the nonlinear rational recursive sequence

$$x_{n+1} = \frac{bx_n^2}{1 + x_{n-1}^2}, \tag{1}$$

where b is a positive real number and the initial values x_0, x_1 are positive. They gave the description of the asymptotic behavior of the solutions of (1). Our aim in this note is to investigate the more general difference equation

$$x_{n+1} = x_n^2 f(x_{n-1}), \tag{2}$$

which is a generalization of (1). Our results imply those obtained in [2].

Theorem. *Consider the difference equation (2) where $f : [0, +\infty) \to [0, +\infty)$ is a nonincreasing continuous function. Then we have the following facts:*

(A) If it holds

$$xf(x) < 1, \tag{3}$$

for all $x > 0$, then any solution of (2) is either

(A1) eventually increasing and it tends to $+\infty$, or

(A2) eventually decreasing and it tends to 0.

1991 *Mathematics Subject Classification.* Primary 39A11.

Key words and phrases. Difference equations, asymptotic properties.

Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

In particular if

$$\sup_{x\geq 0}(x^2 f(x)) < +\infty, \tag{4}$$

then (A2) holds.

(B) Assume that the number

$$K_1 := min\{x > 0 : xf(x) = 1\}$$

exists. Then we have the following facts:

(B1) Any solution with $x_1 \leq x_0 < K_1$ tends monotonically to zero and so K_1 is unstable.

(B2) There is no solution which is oscillatory with respect to K_1.

(C) Assume that the number

$$K_2 := max\{x > 0 : xf(x) = 1\}$$

exists and Ineq. (3) holds for all $x > K_2$. Then we have the following facts:

(C1) There is no solution (x_n) with $x_n > K_2$ eventually for all n.

(C2) If the number K_2 is isolated and the inequality $xf(x) > 1$ holds for all $x < K_2$ and close to K_2, relatively to K_1, then there is no solution which is eventually increasing and tends to K_2.

2. Proof of the Theorem

(A) First of all we observe that, if (3) holds, then the only rest point of equation is 0.

Let (x_n) be a solution of (1) not satisfying (A1). Then there is an index m such that $x_m < x_{m-1}$. Thus from (2) we get

$$x_{m+1} = x_m x_m f(x_{m-1}) \leq x_m x_m f(x_m) < x_m.$$

By induction we conclude that (x_n) is eventually decreasing (and positive). Thus the limit l of (x_n) exists and it is a rest point of (2), thus $l = 0$.

Next assume that (4) is true and let M be an upper bound of $x^2 f(x)$. Then any solution (x_n) satisfies the relation

$$\begin{aligned} x_{n+1} &= x_n^2 f(x_{n-1}) = x_{n-1}^4 [f(x_{n-2})]^2 f(x_{n-1}) \\ &= x_{n-1}^2 x_{n-1}^2 [f(x_{n-2})]^2 f(x_{n-1}) \\ &= [x_{n-1}^2 f(x_{n-1})][x_{n-2}^2 f(x_{n-2})]^2 [f(x_{n-3})]^2 \leq M^3 [f(0)]^2 \end{aligned}$$

for each index $n \geq 3$ and therefore it is bounded. Thus (A1) does not hold.

(B1) For each (x_n) with $x_1 \leq x_0 < K_1$ we have

$$x_2 = x_1 x_1 f(x_0) \leq x_1 x_1 f(x_1) < x_1$$

and by induction

$$x_{n+1} < x_n < K_1.$$

Hence (x_n) is (strictly) decreasing and therefore it tends to 0.

(B2) If for a certain solution (x_n) there is an index m such that

$$x_m < K_1 < x_{m-1},$$

then it holds

$$x_{m+1} = x_m x_m f(x_{m-1}) < x_m K_1 f(K_1) = x_m < K_1$$

and so, as in (B1), $x_{n+1} < x_n < K_1$, for all $n \geq m$. This argument says that (x_n) can not by oscillatory.

(C1) Assume that (2) admits a bounded solution (x_n) such that $x_n > K_2$, for all large n. As there is no rest point of (2) greater than K_2, we conclude that (x_n) is not eventually increasing. Thus for a certain index m we have $x_m < x_{m-1}$. Then

$$x_{m+1} = x_m x_m f(x_{m-1}) \leq x_m x_m f(x_m) < x_m,$$

and by induction $(K_2 <) x_{n+1} < x_n$, for all $n \geq m$. This means that

$$lim x_n =: l \geq K_2$$

exists. Now observe that

$$\frac{x_{n+2}}{x_{n+1}} = x_{n+1} f(x_n) < \frac{x_{n+1}}{x_n} (< 1)$$

and so the

$$lim \frac{x_{n+1}}{x_n} =: \zeta (< 1)$$

exists. Thus we get

$$1 = \frac{l}{l} = \zeta < 1,$$

a contradiction.

(C2) Assume that (2) admits a solution (x_n) which is eventually increasing and it tends to K_2. Then for all large n we have

$$x_n < x_{n+1} < K_2 \text{ and } x_n f(x_n) < x_n.$$

Thus

$$0 < lim x_n =: l \leq K_2$$

exists. Now observe that

$$\frac{x_{n+1}}{x_{n+2}} = \frac{1}{x_{n+1} f(x_n)} = \frac{x_n}{x_{n+1}} \frac{1}{x_n f(x_n)} < \frac{x_n}{x_{n+1}} (< 1)$$

and so the

$$lim \frac{x_n}{x_{n+1}} =: \eta (< 1)$$

exists. Thus we get

$$1 = \frac{l}{l} = \eta < 1,$$

a contradiction.

Remark: One can see that for the function $f(x) := \frac{b}{1+x^2}$ our theorem applies easily to equation (1) and gives the results obtained in [2].

References

[1] V. L. Kocic and G. Ladas, *Global Behavior of Nonlinear Difference Equations of Higher Order and Applications*, Kluwer Academic Publishers, Dordrecht, 1993.

[2] D. C. Zhang, B. Shi and M. J. Gai, *On the rational recursive sequence* $x_{n+1} = bx_n^2/(1+x_{n-1}^2)$, Indian J. Pure Appl. Math. **32(5)** (2001), 657-663.

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